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An Alternant Criterion for Haar Cones

PETRA PEISKER

Abteilung für Mathematik,
Ruhr-Universität Bochum, D–4630 Bochum, West Germany

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A characterization of the best Chebyshev approximation in terms of alternants with sign conditions is established for arbitrary Haar cones.

1. INTRODUCTION

When the Chebyshev approximation by γ polynomials is considered, the local best approximations can be characterized in terms of alternants with signs. This was verified in [2, Part II] by making use of the special properties of those Haar cones which occur as tangent cones for γ polynomials.

In this paper we shall establish such an alternant criterion for arbitrary Haar cones. We note that alternants with sign conditions occur in cases where boundaries are present. In particular similar characterizations have been recently used in the (quite different) investigation of best perfect splines [4].

2. PRELIMINARIES

Let $[a, b]$ be a compact interval and let the space of continuous, real valued functions $C[a, b]$ be endowed with the uniform norm. An n-dimensional linear subspace $H \subset C[a, b]$ is called a Haar space of dimension n, if each nontrivial function $v \in H$ has at most $n-1$ zeros in $[a, b]$. A basis $\{h_1, h_2,..., h_n\}$ of a Haar space is said to be a Haar system. Let $\{h_1, h_2,..., h_n\} \subset C[a, b]$. Then

$$K = \left\{ \sum_{i=1}^{n} \alpha_i h_i;\ \alpha_i \in \mathbb{R},\ \alpha_i \geqslant 0 \text{ for } m+1 \leqslant i \leqslant n \right\}$$

is called a Haar cone of dimension n, if for each subset L, $\{1, 2,..., m\} \subset L \subset$

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$\{1, 2, \ldots, n\}$, the system $\{h_i; i \in L\}$ is a Haar system. The cone K is a proper Haar cone, if $m < n$. To each $u \in K$ represented by $u = \sum_{i=1}^{n} \alpha_i(u) h_i$ is assigned an indexing set

$$J(u) := \{1, 2, \ldots, m\} \cup \{i \geqslant m+1; \alpha_i(u) > 0\}$$

an integer

$$k(u) := |J(U)|$$

and a (possibly larger) Haar cone

$$K_{J(u)} := \left\{ \sum_{i=1}^{n} \alpha_i h_i; \alpha_i \in \mathbb{R}, \alpha_i \geqslant 0 \text{ for } i \notin J(u) \right\}.$$

Furthermore, to each Haar cone K we assign the integer

$$r = r(K) := \sup\{k \in \mathbb{N}; \text{there exists a nontrivial function } v \in K \text{ with } k-1 \text{ zeros in } [a, b]\},$$

which is called the "root-number" of K. In case $m < n$ (i.e., K is a proper cone), we obviously have $m + 1 \leqslant r \leqslant n$.

In the next section we shall refer to

Remark 2.1 ([3, p. 22]). Let $\{1, 2, \ldots, m\} \subset \{l_1, l_2, \ldots, l_k\} \subset \{1, 2, \ldots, n\}$. Then the determinants

$$U\begin{pmatrix} l_1, l_2, \ldots, l_k \\ t_1, t_2, \ldots, t_k \end{pmatrix} := \det(h_{l_i}(t_j))_{i,j=1}^{k}$$

preserve a single strict sign for all choices

$$a \leqslant t_1 < t_2 < \cdots < t_k \leqslant b.$$

Indeed, the Haar condition yields

$$U\begin{pmatrix} l_1, l_2, \ldots, l_k \\ t_1, t_2, \ldots, t_k \end{pmatrix} \neq 0$$

whenever $a \leqslant t_1 < t_2 < \cdots < t_k \leqslant b$. Thus we obtain the statement by virtue of the intermediate value theorem.

3. Properties of Haar Cones

Let

$$K = \left\{ \sum_{i=1}^{n} a_i h_i;\, a_i \in \mathbb{R},\, a_i \geqslant 0 \text{ for } m+1 \leqslant i \leqslant n \right\} \subset C[a, b]$$

be a proper Haar cone of dimension n. Then

$$H_m := \operatorname{span}\{h_1, h_2, \ldots, h_m\}$$

is a proper subset of K. Besides the root number $r = r(K)$ we want to associate a sign $\sigma = \sigma(K)$ to the cone K such that each function $v \in K$ with exactly $r-1$ zeros $a \leqslant t_1 < t_2 < \cdots < t_{r-1} < b$ will attain the sign σ on the right, i.e., $\sigma v(t) > 0$ for $t > t_{r-1}$. This is illustrated by an example.

Example. The set of polynomials with nonnegative highest coefficient

$$K = \left\{ \sum_{\nu=0}^{n} a_\nu x^\nu;\, a_\nu \in \mathbb{R},\, a_n \geqslant 0 \right\}$$

is a proper Haar cone of dimension $n+1$. Here we have $r = n+1$ and each polynomial in K with n roots has the sign $\sigma = +1$ on the right, as may be seen from the asymptotic behavior for $x \to \infty$.

In the general case we only know the functions in the interval $[a, b]$. Therefore the consideration of the asymptotic behavior is not possible. Another approach is necessary.

Proposition 3.1. *Let $l \in \mathbb{N}$, $m+1 \leqslant l \leqslant r$. Then there exists an l-dimensional Haar subcone K_l, $H_m \subset K_l \subset K$, and a function $v \in K_l$ with exactly $l-1$ (distinct) zeros.*

Proof. By definition of $r = r(K)$, the set

$$M_l := \{v \in K;\, v \neq 0,\, v \text{ has at least } l-1 \text{ zeros in } [a, b]\}$$

is not empty. Choose a function v_0 from M_l with minimal $k(v_0)$, and denote the first $l-1$ zeros of v_0 by $s_1 < s_2 < \cdots < s_{l-1}$.

First, from the Haar condition we have

$$k(v_0) \geqslant l.$$

We claim $k(v_0) = l$. Assume to the contrary that $k(v_0) > l$. Let L, $\{1, \ldots, m\} \subset L \subset J(v_0)$, be a set of order l. Since $H_L := \operatorname{span}\{h_i;\, i \in L\}$ is a Haar space of dimension l, there exists a nontrivial function $h \in H_L$ vanishing at the points $s_1 < s_2 < \cdots < s_{l-1}$. With $I := \{1, \dots, m\}$ we write

$$h = \sum_{i=1}^{m} \beta_i h_i + \sum_{\nu \in L \setminus I} \beta_\nu h_\nu$$

$$v_0 = \sum_{i=1}^{m} \alpha_i h_i + \sum_{\nu \in L \setminus I} \alpha_\nu h_\nu + \sum_{\mu \in J(v_0) \setminus L} \alpha_\mu h_\mu .$$

In the case $\beta_\nu \geqslant 0$ for $\nu \in L \setminus I$, we have $h \in K$ contradicting the fact $k(h) < k(v_0)$. In the other case, recalling $\alpha_\nu > 0$ for $\nu \in L \setminus I$, we conclude that there is a constant $\lambda > 0$ such that

$$\alpha_\nu + \lambda \beta_\nu \geqslant 0 \qquad \text{for} \quad \nu \in L \setminus I$$

and

$$\alpha_{\nu_0} + \beta_{\nu_0} = 0 \qquad \text{for at least one} \quad \nu_0 \in L \setminus I.$$

Thus we have $v_0 + \lambda h \in M_l$ and $k(v_0 + \lambda h) < k(v_0)$, which is a contradiction to the choice of v_0. This proves $k(v_0) = l$ and

$$K_l := \left\{ \sum_{i \in J(v_0)} \alpha_i h_i ; \alpha_i \geqslant 0 \text{ for } i \geqslant m+1 \right\}$$

is a cone as stated. ∎

The properties of the subcones K_l just constructed are studied in

PROPOSITION 3.2. *Let $m+1 \leqslant l \leqslant r$ and $H_m \subset K_l \subset K$ be a Haar cone of dimension l. If K_l contains a function with exactly $l-1$ zeros, then there are nontrivial functions in K_l exhibiting $l-1$ prescribed zeros. Moreover all functions in K_l with $l-1$ zeros in (a, b) attain a fixed sign $\sigma = \sigma(K_l)$ on the right-hand side of the zeros.*

Proof. Let K_l be generated by

$$h_1, h_2, \dots, h_m, h_{\nu_{m+1}}, \dots, h_{\nu_l},$$

and let $v_0 \in K_l$ be a function vanishing at $l-1$ points $s_1 < s_2 < \cdots < s_{l-1}$. Since in l-dimensional Haar spaces functions with $l-1$ prescribed zeros are determined up to constants, there is $\lambda_0 \in \mathbb{R}$ such that

$$v_0(t) = \lambda_0 U(t) := \lambda_0 U \begin{pmatrix} 1, \dots, m, \nu_{m+1}, \dots, \nu_{l-1}, \nu_l \\ s_1, \qquad \cdots \qquad , s_{l-1}, t \end{pmatrix}.$$

We claim that

$$\lambda_0 U \begin{pmatrix} 1,\ldots, m, \nu_{m+1},\ldots, \nu_{l-1}, \nu_l \\ t_1, \quad \cdots \quad , t_{l-1}, t \end{pmatrix} \in K_l \tag{$*$}$$

for all choices $a \leqslant t_1 < t_2 < \cdots < t_{l-1} \leqslant b$.

Indeed, by expanding determinant $(*)$ along the last column, we conclude from Remark 2.1 that the coefficients associated with the functions $h_{\nu_{m+1}}$, $h_{\nu_{m+2}},\ldots, h_{\nu_l}$ have the same sign as the corresponding coefficients of $\lambda_0 U(t)$. Furthermore, again by Remark 2.1, all functions of form $(*)$ have the same sign $=: \sigma$ for $t > t_{l-1}$. Thus we get the statement concerning the sign by recalling that in l-dimensional proper Haar cones functions vanishing at $l-1$ points are unique up to positive constants. ∎

As an immediate consequence of the Propositions 3.1 and 3.2 we obtain

COROLLARY 3.3. *All functions belonging to the proper Haar cone K and vanishing at exactly $r-1$ points, $r = r(K)$, exhibit a uniform sign on the right-hand side of the zeros.*

Proof. Note that the cones specified in Proposition 3.1 are not unique. Let K_r and $\tilde{K}_r$ be cones in the sense of that proposition. The proof is complete if we verify $\sigma(K_r) = \sigma(\tilde{K}_r)$. Indeed, let $v \in K_r$, $\tilde{v} \in \tilde{K}_r$ be functions with the same given zeros $t_1 < t_2 < \cdots < t_{r-1}$. If v and $\tilde{v}$ attain the opposite sign for $t > t_{r-1}$, then a convex combination has an additional zero. Now the case of arbitrary functions in K with $r-1$ zeros (not necessarily belonging to r-dimensional subcones) is treated in an obvious way. ∎

This sign specified in the corollary will be denoted as $\sigma(K)$. Furthermore, we shall denote by $K_{l,\varepsilon}$, $H_m \subset K_{l,\varepsilon} \subset K$, a Haar cone of dimension l containing a function with exactly $l-1$ zeros and the sign ε on the right-hand side. This notation is justified by virtue of Proposition 3.2. Note that for $l = r(K)$ only the sign $\sigma(K)$ is possible. In case $l < r$, however, both signs are exhibited.

PROPOSITION 3.4. (Existence of subcones with prescribed sign). *Let $m+1 \leqslant l < r$ and $\varepsilon \in \{-1, 1\}$. Then there exists a subcone $K_{l,\varepsilon}$.*

Proof. According to Proposition 3.1 there is a $(l+1)$-dimensional Haar cone K_{l+1} containing a function v with l zeros $a \leqslant s_1 < s_2 < \cdots < s_l < b$. Since the zeros of v are all simple, there are $\tau_1, \tau_2 \in (s_{l-1}, b)$ such that

$$v(\tau_1) < 0 \qquad \text{and} \qquad v(\tau_2) > 0.$$

Consequently, the set

$$M_{l,\varepsilon} := \{v \in K; v \text{ has (at least) } l-1 \text{ zeros } a \leqslant t_1 < t_2 < \cdots < t_{l-1} < b \text{ and there is } \tau \in (t_{l-1}, b) \text{ such that } \varepsilon v(\tau) > 0\}$$

is not empty. Choose $v_0 \in M_{l,\varepsilon}$ with minimal $k(v_0)$. As in the proof of Proposition 3.1 we conclude $k(v_0) = l$. Thus v_0 has exactly $l-1$ zeros and the sign ε on the right-hand side. Hence,

$$K_{l,\varepsilon} := \left\{ \sum_{i \in J(v_0)} \alpha_i h_i; \alpha_i \geqslant 0 \text{ for } i \geqslant m+1 \right\}$$

is a cone as stated. ∎

4. The Alternant Criterion

As a consequence of the properties of Haar cones established in the previous section, we shall obtain a characterization of best approximations in terms of alternants with signs. The uniqueness of best approximations has already been proved in [1].

Let K be a Haar cone of dimension n. Since an alternant criterion for Haar spaces is well known, we may restrict our attention to the case where K is a proper cone. Then, by virtue of Section 3, a tuple (r, σ) is assigned to K. Since K is convex, $v \in K$ is the best approximation to $f \in C[a,b]$ in K, if and only if zero is the best approximation of $f - v$ in the Haar cone $K_{J(v)}$ [1]. Thus, after replacing K by $K_{J(v)}$, we may assume $v = 0$.

Theorem 4.1. *Let K be a proper Haar cone and (r, σ) the associated tuple. Then the following properties are equivalent:*

(i) *Zero is the best approximation to f in K.*

(ii) *There is an alternant to f of length r with sign $-\sigma$ on the right.*

Proof. (a) Assume that there is an alternant of length r with the sign $-\sigma$ on the right, i.e., there are points $a \leqslant t_1 < t_2 < \cdots < t_r \leqslant b$ such that

$$(-\sigma)(-1)^{r-i} f(t_i) = \|f\|, \qquad 1 \leqslant i \leqslant r.$$

Suppose that there is a better approximation $v \in K$. Then for $1 \leqslant i \leqslant r$ we have

$$(-\sigma)(-1)^{r-i}(f - v)(t_i) \leqslant \|f - v\| < \|f\| = (-\sigma)(-1)^{r-i} f(t_i).$$

This implies

$$(-1)^{r-i}(-\sigma)\, v(t_i) > 0, \qquad i = 1, 2, \ldots, r.$$

Hence $v \in K$ has $r-1$ zeros and the "wrong" sign $-\sigma$ on the right, which is a contradiction.

(b) Let zero be the best approximation to f in K. Assume that there is no alternant to f of length r with sign $-\sigma$ on the right. Let l denote the maximum length of an alternant to f and ε the associated sign. Then we have either $l < r$ or $l = r$, $\varepsilon = \sigma$. Since K contains a Haar subspace of dimension m, we have $l \geqslant m + 1$.

According to Proposition 3.4 (or Proposition 3.1) there is a Haar subcone $K_{l,\varepsilon}$. From the classical characterization for best approximations in linear spaces we know that the l-dimensional Haar space $H := \operatorname{span}(K_{l,\varepsilon})$ contains a function $h \in H$ such that

$$\|f - h\| < \|f\|.$$

With the same arguments as above it follows that

$$\varepsilon(-1)^{l-i}\, h(t_i) > 0, \qquad i = 1, 2, \ldots, l.$$

if $t_1 < t_2 < \cdots < t_l$ are the points of the alternant. Thus the function h has $l-1$ zeros and the sign ε on the right, which means that h belongs to the subset $K_{l,\varepsilon} \subset H$, contradicting the optimality of zero. ∎

In [2, Part II] the best approximations in the tangent cones of sets of γ polynomials are characterized by alternants. There the length of the alternant is expressed in a technical way. From Theorem 4.1 the meaning of this number becomes more transparent; see also [6].

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